Asymptotics of implied volatility in local volatility models

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ASYMPTOTICS OF IMPLIED VOLATILITY IN LOCAL VOLATILITY MODELS

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ABSTRACT. Using an expansion of the transition density function of a 1-dimensional time inhomogeneous diffusion, we obtain the first and second order terms in the short time asymptotics of European call option prices. The method described can be generalized to any order. We then use these option prices approximations to calculate the first order and second order deviation of the implied volatility from its leading value and obtain approximations which we numerically demonstrate to be highly accurate. The analysis is extended to degenerate diffusions using probabilistic methods, i.e. the so called principle of not feeling the boundary.

CONTENTS

1. INTRODUCTION

Stochastic volatility models offer a widely accepted approach to incorporating into the modeling of option markets a flexibility that accounts for the implied volatility smile or skew, see Gatheral [14] for a in depth introduction to the subject. Historically, the first models to be introduced into the literature were the Hull and White model in [22], the Stein and Stein model in [31], and the Heston model in [20]. In these three models, the underlying asset and its volatility are driven by Brownian motions that may be instantaneously correlated. The correlation coefficient is taken to be a constant. Later on Bates [3] introduced the first of a series of models incorporating jumps, and these were followed by Andersen and Andreasen [2]. Recent years have seen an explosion of models using the method of stochastic time changes to produce ever more versatile models, see for instance Carr et al [8] and Carr and Wu [10]. However, purely diffusive models have not stopped being popular. A case in point is the introduction into the literature of the SABR (stochastic alpha-beta-rho) model by Hagan, Lesniewski and Woodward [16]:

$$dF_t = F_t^{\beta} y_t dW_{1t},\ dy_t = y_t dW_{2t},\ dW_{1t}dW_{2t} = \rho dt;$$
$$F_0 = \bar{F}_0,\ y_0 = \alpha.$$

This model was generalized by Henry-Labordère, who in [18] introduced the λ-SABR model, in which the second equation is complemented by a mean-reverting term

$$dy_t = \lambda(\theta - y_t)dt + y_t dW_{2t}\ .$$

Hagan and Woodward [15] used perturbation theory to find asymptotic expansions for the implied volatility of European options in a local volatility setting. Then Hagan, Kumar, Lesniewski, and Woodward [17] used asymptotic methods to obtain approximations for the implied volatility in the two factor SABR models. In a Courant Institute lecture, Lesniewski [26] introduced a geometric approach to asymptotics by relating the underlying geometry of the diffusion process associated with the SABR model in the case $\beta = 0$ to the Poincaré upper half plane, a model of hyperbolic space, and outlined an approach to asymptotics in stochastic volatility models via a WKB expansion. This approach was further developed in an important unpublished working paper by Hagan, Lesniewski, and Woodward [16]. These authors used changes of variables to reduce the SABR model with $\beta \neq 0$ to a perturbed form of the same model with $\beta = 0$ and then used the Hausdorff-Baker Campbell formula to find approximate solutions for the fundamental solution of the perturbed problem.

Henry-Labordère in [18] made contributions of both theoretical and practical nature. As mentioned above, he introduced the λ-SABR model, a two factor stochastic volatility model with a mean reverting drift term for the volatility and showed how the heat kernel method yields asymptotic formulas for the fundamental solution and for the implied volatility and local volatility in this model. An analogous result based on the stochastic framework in Molchanov [29] was applied in a working paper by Bourgade and Croissant [6] to a homogeneous version of the SABR model.

In the present paper we focus on the *local volatility model* and reconsider the asymptotic expansion of implied volatility for small time to maturity. In the case when the volatility *does not depend explicitly on time* (i.e., time homogeneous models), our result's leading order (zeroth order) term agrees with those of Berestycki, Busca, and Florent[4], Hagan, Lesniewski, and Woodward[16], and Henry-Labordère[18]. When the local volatility depends explicitly on time, i.e., for *time inhomogeneous models*, we find that even the formula for the zeroth order term requires a small but key correction. In the first order and still in the case of time independent volatility, our formula is different from and more accurate than the ones in [15], [16], [17], and [18]. In fact we show rigorously that our first order correction really is the first order derivative of the implied volatility with respect to the time to maturity. This characterization alone only implies that for very small time to maturity the formula is optimal. If the added accuracy brought by the formula was only for very small times this would limit its usefulness. Our numerical experiments show that there is a gain in accuracy for a wide range of times. After this work was completed, it was called to our attention that our first order correction, in the time homogeneous case and when $r = 0$, had already been discovered by Henry-Labordère and appeared into his book [19]. There it is obtained by a heuristic procedure, in which the nonlinear equation of Berestycki et al [4] satisfied by the implied volatility is expanded in powers of the time to maturity. Our approach allows us to rigorously justify this formula. The form of the first order correction we give in the case $r \neq 0$ appears to be new, as does the formula for the σ_1 in the case of time inhomogeneous diffusions. In the simplest case $r = 0$ and time homogeneous diffusions the first order correction is given by

$$\hat{\sigma}_1 = \frac{\hat{\sigma}_0^3}{(\ln K - \ln s)^2} \ln \frac{\sqrt{\sigma(s)\sigma(K)}}{\hat{\sigma}(T,s)},$$

where

$$\hat{\sigma}_0 = \left[\frac{1}{\ln K - \ln s} \int_s^K \frac{du}{u\sigma(u)} \right]^{-1}$$

is the leading term of the implied volatility.

In this paper we take the analysis one step further and determine, in addition to the first order correction, the second order correction σ_2 in the case $r = 0$, but with time dependent coefficients. The second order correction in the case $r \neq 0$ can then readily be obtained by a procedure we will briefly describe.

Although our main objective in this paper is to refine the asymptotics in the local volatility setting, as was shown in the working paper by Hagan, Lesniewski, and Woodward and then by Henry-Labordère, one may use a two-step method discovered independently by Gyöngy and Dupire to derive an implied volatility in the SABR or λ-SABR model by first obtaining an equivalent local volatility model, using Laplace asymptotics, and then using the implied volatility for the resulting model. This will be developped in a forthcoming paper.

Besides the results given here, the only other rigorous results leading to a justification of the zeroth order approximation in local and stochastic volatility models we are aware of were provided by Berestycki, Busca, and Florent in [4] and [5]. Medvedev and Scaillet [28] and Henry-Labordère in his book [19] have shown how to obtain the results by matching the coefficients of the powers of the time to expiration in the nonlinear partial differential equation satisfied by the implied volatility. Also Kunimoto and Takahashi began in [24] a series of papers which bases a rigorous perturbation theory on Malliavin-Watanabe calculus. Takahashi and collaborators have recently applied this approach to the $\lambda-$Sabr model [32].

We derive our results by two different methods. The first is a probabilistic approach. Since it is technically rather simple, we provide the full detail of the proof. This approach is described in SECTION 2. The second approach, described much more briefly in SECTION 3, is via the "geometric expansion" coupled with the Levi parametrix method. This approach was first discovered by Yoshida [34], but modern references seem to have overlooked his contribution. Some additional details and a very clear exposition can be found in Chavel

[11]. In SECTION 2 we only carry the expansion out to order one. It turns out that the second method in SECTION 3 is computationally quite efficient, so we use it to compute the first and second order corrections for the call prices and to compute the first order correction in the time inhomogeneous case as well as the second order correction. A numerical section, SECTION **??**, explores the effectiveness of the expansions obtained and compares these to earlier expansions.

2. PROBABILISTIC APPROACH

2.1. **Call price expansion.** Suppose that the dynamics of the stock price S is given by

$$dS_t = S_t \left\{ r\,dt + \sigma(S_t)\,dW_t \right\}.$$

Then the stochastic differential equation for the logarithmic stock price process $X = \ln S$ is

$$dX_t = \eta(X_t)dW_t - \frac{1}{2}\eta^2(X_t)dt + rdt,$$

where $\eta(x) = \sigma(e^x)$. Denote by $c(t,s)$ the price of the European call option (with expiry T and strike price K understood) at time t and stock price s in the local volatility model. It satisfies the following Black-Scholes equation

$$c_t + \frac{1}{2}\sigma(s)^2 s^2 c_{ss} + rsc_s - rc = 0, \quad c(T,s) = (s-K)^+.$$

It is more convenient to work with the function

$$v(\tau,x) = c(T-\tau, e^x), \quad (\tau,x) \in [0,T] \times \mathbb{R}.$$

The reason is that for this function the Black-Scholes equation takes a simpler form

$$v_\tau = \frac{1}{2}\eta^2(x)v_{xx} + \left[r - \frac{1}{2}\eta(x)^2 \right] v_x - rv, \quad v(0,x) = (e^x - K)^+. \tag{2.1}$$

We now study the the asymptotic behavior of the modified call price function $v(\tau,x)$ as $\tau \downarrow 0$. Our basic technical assumption is as follows. There is a positive constant C such that for all $x \in \mathbb{R}$,

$$C^{-1} \le \eta(x),\ |\eta'(x)| \le C,\ |\eta''(x)| \le C.$$

Since $\eta(x) = \sigma(e^x)$, the above assumption is equivalent to the assumption that there is a constant C such that for all $s \ge 0$,

$$C^{-1} \le \sigma(s) \le C,\ |s\sigma'(s)| \le C,\ |s^2\sigma''(s)| \le C. \tag{2.2}$$

For many popular models (e.g., CEV model), these conditions may not be satisfied in a neighborhood of the boundary points $s = 0$ and $s = \infty$. However, since we only consider the situation where the stock price and the strike have fixed values other than these boundary values, or more generally vary in a bounded closed subinterval of $(0, \infty)$, the behavior of the coefficient functions in a neighborhood of the boundary points will not affect the asymptotic expansions of the transition density and the call price. Therefore we are free to modify the values of σ in a neighborhood of $s = 0$ and $s = \infty$ so that the above conditions are satisfied. The principle of of not feeling the boundary (see APPENDIX A) shows that such modification only produce a exponentially negligible error which will not show up in the relevant asymptotic expansions.

Proposition 2.1. *Let $X = \ln S$ be the logarithmic stock price. Denote the density function of X_t by $p_X(\tau, x, y)$. Then as $\tau \downarrow 0$,*

$$p_X(\tau, x, y) = \frac{u_0(x, y)}{\sqrt{2\pi\tau}} e^{-\frac{d^2(x,y)}{2\tau}} \left[1 + O(\tau; x, y)\right].$$

Here

$$d(x, y) = \int_x^y \frac{du}{\eta(u)}$$

and

$$u_0(x, y) = \eta(x)^{1/2}\eta(y)^{-3/2} \exp\left[-\frac{1}{2}(y - x) + r\int_x^y \frac{du}{\eta(u)^2}\right]. \tag{2.3}$$

Furthermore, the remainder satisfies the inequality $|O(\tau; x, y)| \le C\tau$ for some constant independent of x and y.

Proof. See PROPOSITION 2.6 in SECTION 2.3. □

Now we compute the leading term of $v(\tau, x)$ as $\tau \downarrow 0$. For the heat kernel $p_X(\tau, x, y)$ itself, the leading term is

$$\frac{u_0(x, y)}{\sqrt{2\pi\tau}} \exp\left[-\frac{d(x, y)^2}{2\tau}\right].$$

The in the money case $s > K$ yielding nothing extra (see SECTION 2.4), we only consider the out of the money case $s < K$, or equivalently, $x < \ln K$. We express the modified price function $v(\tau, x)$ in terms of the density function $p_X(\tau, x, y)$ of X_t. We have

$$\begin{aligned} v(\tau, x) =& c(T-\tau, e^x) \\ =& e^{-r\tau}\mathbb{E}\left[(S_T - K)^+ | S_{T-\tau} = e^x\right] \\ =& e^{-r\tau}\mathbb{E}\left[(S_\tau - K)^+ | S_0 = e^x\right] \\ =& e^{-r\tau}\mathbb{E}_x\left[(e^{X_\tau} - K)^+\right]. \end{aligned}$$

In the third step we used the Markov property of S. Therefore

$$\begin{aligned} v(\tau, x) &= \frac{1}{\sqrt{2\pi\tau}} \int_{\ln K}^{\infty} (e^y - K) e^{-r\tau} p_X(\tau, x, y)\, dy \\ &= \frac{1}{\sqrt{2\pi\tau}} \int_{\ln K}^{\infty} (e^y - K) e^{-r\tau} u_0(x, y) \\ &\qquad \exp\left[-\frac{d(x,y)^2}{2}\right] (1 + O(\tau; x, y))\; dy. \end{aligned} \tag{2.4}$$

From the inequality $|O(\tau; x, y)| \le C\tau$ for some constant C, it is clear from (2.4) that remainder term will not contribute to the leading term of $v(\tau, x)$. For this reason we will ignore it completely in the subsequent calculations. Similarly, because $|e^{-r\tau} - 1| \le r\tau$, we can replace $e^{-r\tau}$ in the integrand by 1. A quick inspection of (2.4) reveals that the leading term is determined by the values of the integrand near the point $y = \ln K$. Introducing the new variable $z = y - \ln K$ and letting $\rho = \ln K - x$, we conclude that $v(x, \tau)$ has the same leading term as the function

$$\begin{aligned} v^\sharp(\tau, x) &= \frac{K}{\sqrt{2\pi\tau}} \int_0^{\infty} (e^z - 1) u_0(x, z + \ln K) \\ &\qquad \exp\left[-\frac{d(x, z + \ln K)^2}{2\tau}\right] dz. \end{aligned} \tag{2.5}$$

The key calculation is contained in the proof of the following result.

Lemma 2.2. *We have* $\tau \downarrow 0$

$$\begin{aligned} &\int_0^{\infty} z^k \exp\left[-\frac{d(x, z+\ln K)^2}{2\tau}\right] dz \\ &\qquad \sim k! \left[\frac{\sigma(K)\tau}{d(x, \ln K)}\right]^{k+1} \exp\left[-\frac{d(x, \ln K)^2}{2\tau}\right]. \end{aligned}$$

Proof. See LEMMA 2.7 in SECTION 2.3. We only need the cases $k = 1$ and 2. □

The main result of this section is the following.

Theorem 2.3. *Suppose that the volatility function σ satisfies the basic assumption (2.2). If $x < \ln K$ we have as $\tau \downarrow 0$,*

$$v(\tau, x) \sim \frac{Ku_0(x, \ln K)}{\sqrt{2\pi}} \left[\frac{\sigma(K)}{d(x, \ln K)}\right]^2 \tau^{3/2} e^{-\frac{d(x,\ln K)^2}{2\tau}}.$$

Proof. We have shown that $v(x, \tau)$ and $v^\sharp(x, \tau)$ defined in (2.5) has the same leading term. We need to replace the function before the exponential factor by its value at the boundary point $z = 0$. First of all, we have

$$|e^z - 1 - z| \le z^2 e^z$$

From the explicit expression (2.3) and the basic assumption (2.2) it is easy to verify that

$$|u_0(x, z + \ln K) - u_0(x, \ln K)| \le ze^{Cz}$$

for some positive constant C. By the estimate in LEMMA 2.2 we obtain

$$\begin{aligned} v(\tau, x) &\sim v^\sharp(\tau, x) \\ &\sim \frac{Ku_0(x, \ln K)}{\sqrt{2\pi\tau}} \int_0^\infty z \exp\left[-\frac{d(x, z + \ln K)^2}{2\tau}\right] dz \\ &\sim \frac{Ku_0(x, \ln K)}{\sqrt{2\pi\tau}} \left[\frac{\sigma(K)\tau}{d(x, \ln K)}\right]^2 e^{-\frac{d(x,\ln K)^2}{2\tau}}. \end{aligned}$$

□

2.2. **Implied volatility expansion.** Using the leading term of the call price function calculated in the previous section, we are now in a position to prove the main theorem on the asymptotic behavior of the implied volatility $\hat{\sigma}(t, s)$ near expiry T. We will obtain this by comparing the leading terms of the relation

$$c(t, s) = C(t, s; \hat{\sigma}(t, s), r).$$

Here $C(t, s; \sigma, r)$ is the classical Black-Scholes pricing function. For this purpose, we need to calculate the leading term of the classical Black-Scholes call price function. Our main result is the following.

Theorem 2.4. *Let $\hat{\sigma}(t, s)$ be the implied volatility when the stock price is s at time t. Then we have near the expiry T,*

$$\hat{\sigma}(t, s) = \hat{\sigma}(T, s) + \hat{\sigma}_1(T, s)(T - t) + O((T - t)^2),$$

where

$$\hat{\sigma}(T, s) = \left[\frac{1}{\ln K - \ln s} \int_s^K \frac{du}{u\sigma(u)}\right]^{-1}$$

and $\hat{\sigma}_1(T,s)$ *is given by*

$$\text{(2.6)}\quad \frac{\hat{\sigma}^(T,s)^3}{(\ln K - \ln s)^2}\left[\ln\frac{\sqrt{\sigma(s)\sigma(K)}}{\hat{\sigma}(T,s)} + r\int_s^K \left(\frac{1}{\sigma^2(u)} - \frac{1}{\hat{\sigma}^2(T,s)}\right)\frac{du}{u}\right].$$

As we have mentioned above, the leading term $\hat{\sigma}(T,s)$ was obtained in Berestycki, Busca and Florent [4]. They first derived a quasi-linear partial differential equation for the implied volatility and used a comparison argument. When there interest rate $r = 0$, the first order approximation of the implied volatility is given by

$$\hat{\sigma}_1(T,s) = \frac{\hat{\sigma}^(T,s)^3}{(\ln K - \ln s)^2}\ln\frac{\sqrt{\sigma(s)\sigma(K)}}{\hat{\sigma}(T,s)}. \tag{2.7}$$

This case has already appeared in Henry-Labordère [19]. To begin we establish the following

Lemma 2.5. *Let* $V(\tau,x;\sigma,r) = C(T-\tau,e^x;\sigma,r)$ *be the classical Black-Scholes call price function. Then we have as* $\tau\downarrow 0$,

$$V(\tau,x;\sigma,r) \sim \frac{1}{\sqrt{2\pi}}\frac{K\sigma^3\tau^{3/2}}{(\ln K - x)^2}\exp\left[-\frac{\ln K - x}{2} + \frac{r(\ln K - x)}{\sigma^2}\right]$$
$$\exp\left[-\frac{(\ln K - x)^2}{2\tau\sigma^2}\right] + R(\tau,x;\sigma,r).$$

The remainder satisfies

$$|R(\tau,x;\sigma,r)| \le C\,\tau^{5/2}\exp\left[-\frac{(\ln K - x)^2}{2\tau\sigma^2}\right],$$

where $C = C(x,\sigma,r,K)$ *is uniformly bounded if all the indicated parameters vary in a bounded region.*

Proof. The result can be proved starting from the classical Black-Scholes formula for $V(\tau,x;\sigma,r)$. We omit the detail. Note that the leading term can also be obtained directly from THEOREM 2.3 by assuming that $\sigma(x)$ is independent of x. □

We now in a position to complete the proof of the main THEOREM 2.4. Set $\tilde{\sigma}(\tau,x) = \hat{\sigma}(T-\tau,s)$. From the relation

$$v(\tau,x) = V(\tau,x;\tilde{\sigma}(\tau,x),r) \tag{2.8}$$

and their expansions we see that the limit

$$\hat{\sigma}(T,s) = \lim_{\tau\downarrow 0}\tilde{\sigma}(\tau,x)$$

exists and is given by the given expression in the statement of the theorem. Indeed, by comparing only the exponential factors on the two sides we obtain

$$d(x,K) = \frac{\ln K - x}{\tilde{\sigma}(0,x)},$$

from which the desired expression for $\tilde{\sigma}(0,x)$ follows immediately.

Next, let

$$\tilde{\sigma}_1(\tau,x) = \frac{\tilde{\sigma}(\tau,x) - \tilde{\sigma}(0,x)}{\tau}.$$

We have obviously

$$\tilde{\sigma}(\tau,x) = \tilde{\sigma}(0,x) + \tilde{\sigma}_1(\tau,x)\tau. \tag{2.9}$$

From this a simple computation shows that

$$\exp\left[-\frac{(\ln K - x)^2}{2\tau\tilde{\sigma}(\tau,x)^2}\right] = \exp\left[-\frac{d^2}{2\tau} + \frac{\rho^2}{\sigma_0^3}\cdot\sigma_1(\tau,x)\right][1 + O(\tau)], \tag{2.10}$$

where for simplicity we have set

$$\rho = \ln K - x,\ d = d(x,K),\ \sigma_0 = \tilde{\sigma}(0,x)$$

on the right side. From (2.8) and (2.9) we have

$$v(\tau,x) = V\left(\tau,x;\tilde{\sigma}(0,x) + \sigma_1(\tau,x)\tau, r\right).$$

We now use the asymptotic expansions for $v(\tau,x)$ and $V(\tau,x;\sigma,r)$ given by THEOREM 2.3 and LEMMA 2.5, respectively, and then apply (2.10) to the second exponential factor in the equivalent expression for $V(\tau,x;\tilde{\sigma},r)$. After some simplification we obtain

$$u_0\,\sigma(K)^2(1 + O(\tau)) = \sigma_0 \exp\left[-\frac{\rho}{2} + \frac{r\rho}{\sigma_0^2}\right]\exp\left[-\frac{\rho^2}{\sigma_0^3}\cdot\tilde{\sigma}_1(\tau,x)\right],$$

where $u_0 = u_0(x, \ln K)$. Letting $\tau \downarrow 0$, we see that the limit

$$\tilde{\sigma}_1(0,x) = \lim_{\tau\downarrow 0}\frac{\tilde{\sigma}(\tau,x) - \tilde{\sigma}(0,x)}{\tau}$$

exists and is given by

$$\sigma_1(0,x) = \frac{r}{\rho} - \frac{\sigma_0^3}{2\rho^2} - \frac{\sigma_0^3}{\rho^2}\ln\frac{u_0\sigma(K)^2}{\sigma_0}.$$

Using the expression of $u_0 = u_0(x,\ln K)$ in (2.3) we immediately obtain the formula for $\sigma_1(0,x)$.

2.3. **Some computations.** This subsection contains the proofs of the technical results used in the previous subsections. We retain the notation we have used so far. The results are restated for easy reference.

Proposition 2.6. *Let $X = \ln S$ be the logarithmic stock price process. Denote the density function of X_τ by $p_X(\tau, x, y)$. Then we have the following expansion as $\tau \downarrow 0$*

$$p_X(\tau, x, y) = \frac{u_0(x, y)}{\sqrt{2\pi\tau}} e^{-\frac{d(x,y)^2}{2\tau}} \left[1 + O(\tau; x, y)\right],$$

where

$$u_0(x, y) = \eta^{\frac{1}{2}}(x)\eta^{-\frac{3}{2}}(y) \exp\left[-\frac{1}{2}(y - x) + r\int_x^y \frac{du}{\eta(u)^2}\right].$$

For the remainder we have $|O(\tau; x, y)| \le C\tau$ for some constant C independent of x and y.

Proof. Recall that $\{X_t\}$ satisfies the following stochastic differential equation

$$dX_t = \eta(X_t)dW_t - \frac{1}{2}\eta^2(X_t)dt + rdt,$$

where $\eta(x) = \sigma(e^x)$. Introduce the function

$$f(z) = \int_x^z \frac{du}{\eta(u)}$$

and let $Y_t = f(X_t)$. Then

$$dY_t = dW_t - h(Y_t)dt.$$

Here

$$h(y) = \frac{\eta \circ f^{-1}(y) + \eta' \circ f^{-1}(y)}{2} - \frac{r}{\eta \circ f^{-1}(y)}.$$

It is enough to study the the transition density function $p_Y(\tau, x, y)$ of the process Y.

Introduce the exponential martingale

$$Z_\tau = \exp\left[\int_0^\tau h(Y_s)dW_s - \frac{1}{2}\int_0^\tau h^2(Y_s)ds\right]$$

and a new probability measure $\tilde{\mathbb{P}}$ by $d\tilde{\mathbb{P}} = Z_T d\mathbb{P}$. By Girsanov's theorem, the process Y is a standard Brownian motion under $\tilde{\mathbb{P}}$. For any bounded positive measurable function φ we have

$$\int_\Omega \varphi(Y_\tau)d\tilde{\mathbb{P}} = \int_\Omega \varphi(Y_\tau)Z_\tau d\mathbb{P}.$$

Hence, denoting $\mathbb{E}_{x,y}\{\cdot\} = \mathbb{E}_x\{\cdot|Y_\tau = y\}$, we have

$$\int \varphi(y)p(\tau,x,y)dy = \int \varphi(y)\mathbb{E}_{x,y}(Z_\tau)p_Y(\tau,x,y)dy,$$

where

$$p(\tau,x,y) = \frac{1}{\sqrt{2\pi\tau}}\exp\left[-\frac{(y-x)^2}{2\tau}\right]$$

is the transition density function of a standard one dimensional Brownian motion. It follows that

$$\frac{p(\tau,x,y)}{p_Y(\tau,x,y)} = \mathbb{E}_{x,y}(Z_\tau). \tag{2.11}$$

For the conditional expectation of Z_τ, we have

$$\begin{aligned}\mathbb{E}_{x,y}Z_\tau &= \mathbb{E}_{x,y}\exp\left[\int_0^\tau h(Y_s)\circ dY_s - \frac{1}{2}\int_0^\tau \left\{h'(Y_s) - h^2(Y_s)\right\}ds\right]\\ &= e^{H(y)-H(x)}\mathbb{E}_{x,y}\exp\left[-\frac{1}{2}\int_0^\tau h'(Y_s) - h^2(Y_s)ds\right],\end{aligned}$$

where $H'(y) = h(y)$. The relation (2.11) now reads as

$$\frac{p(\tau,x,y)e^{H(y)}}{p_Y(\tau,x,y)e^{H(x)}} = \mathbb{E}_{x,y}\exp\left[-\frac{1}{2}\int_0^\tau\left\{h'(Y_s) - h^2(Y_s)\right\}ds\right]. \tag{2.12}$$

Using the assumption that η and its first and second derivatives are uniformly bounded it is easy to see that the conditional expectation above is of the form $1 + O(\tau;x,y)$ and $|O(\tau;x,y)| \le C\tau$ with a constant C independent of x and y. From

$$h(y) = \frac{\eta\circ f^{-1}(y) + \eta_x\circ f^{-1}(y)}{2} - \frac{r}{\eta\circ f^{-1}(y)}$$

we have

$$H(y) - H(x) = \frac{f^{-1}(y) - f^{-1}(x)}{2} + \frac{1}{2}\ln\frac{\eta(f^{-1}(y))}{\eta(f^{-1}(x))} - \int_{f^{-1}(x)}^{f^{-1}(y)}\frac{r}{\eta^2(v)}dv.$$

This together with (2.12) gives us the asymptotics $p_Y(\tau,x,y)$ of Y_t. Once $p_Y(\tau,x,y)$ is found, it is easy to convert it into the density of $X_t = f^{-1}(Y_t)$ using the formula $p_X(\tau,x,y) = p_Y(\tau,f(x),f(y))f'(y)$. □

Lemma 2.7. *We have*

$$\int_0^\infty z^k \exp\left[-\frac{d(x,z+\ln K)^2}{2\tau}\right] dz \sim k!\left[\frac{\sigma(K)\tau}{d(x,\ln K)}\right]^{k+1} \exp\left[-\frac{d(x,\ln K)^2}{2\tau}\right].$$

Proof. We follow the method in de Bruin [7]. Recall that

$$d(x,y) = \int_x^y \frac{du}{\eta(u)}.$$

Let

$$f(z) = d(x,z+\ln K)^2 - d(x,\ln K)^2.$$

The essential part of the exponential factor is $e^{-f(z)/2\tau}$. For any $\epsilon > 0$, there is $\lambda > 0$ such that $f(z) \geq \lambda$ for all $z \geq \epsilon$, hence

$$\frac{f(z)}{\tau} \geq \left(\frac{1}{\tau} - 1\right)\lambda + f(z).$$

From our basic assumptions (2.2) we see that there is a positive constant C such that $f(z) \geq Cz^2$ for sufficiently large z. Since the integral

$$\int_0^\infty z^k e^{-Cz^2}\, dz$$

is finite, the part of the original integral in the range $[\epsilon, \infty)$ does not contribute to the leading term of the integral. On the other hand, near $z = 0$ we have $f(z) \sim f'(0)z$ with $f'(0) = 2d(x,\ln K)/\sigma(K)$. It follows that the integral has the same leading term as

$$\exp\left[-\frac{d(x,\ln K)^2}{2\tau}\right] \int_0^\infty z^k \exp\left[-\frac{d(x,\ln K)}{\sigma(K)\tau}\, z\right].$$

The last integral can be computed easily and we obtain the desired result. □

2.4. **In the money case.** For the in the money case $s > K$, from

$$(s-K)^+ = (s-K) + (s-K)^-,$$

we have

$$\begin{aligned} & u(\tau,x) - e^x \\ &= \mathbb{E}_x\left[(e^{X_\tau} - K)^+ e^{-r\tau}\right] - e^x \\ &= \mathbb{E}_x\left[e^{X_\tau}e^{-r\tau} - e^x\right] + e^{-r\tau}\mathbb{E}_x\left(e^{X_\tau} - K\right)^- - Ke^{-r\tau}. \end{aligned}$$

The process $e^{X_\tau}e^{-r\tau} = S_\tau e^{-r\tau}$ is a martingale in τ starting from e^x, hence the first term in the right side of the above equation vanishes. The calculation of the leading term of the second term is similar to that of the case when $s < K$. Therefore for $s > K$ we have the following asymptotic expansion

$$u(\tau,x) \sim e^x - Ke^{-r\tau} - \frac{Ku_0(x,\ln K)}{\sqrt{2\pi}}\left[\frac{\sigma(K)}{d(x,\ln K)}\right]^2 \tau^{3/2}\exp\left[-\frac{d(x,\ln K)^2}{2\tau}\right].$$

It is easy to see that this case produces nothing new.

3. Yoshida's approach to heat kernel expansion

3.1. **Time inhomogeneous equations in one dimension.** In this section we review an expression of the heat kernel for a general non-degenerate linear parabolic differential equation due to Yoshida [34]. We will only work out the one dimensional case but in a form that is more general than we actually need in view of possible future use. In our opinion, this form of heat kernel expansion is more efficient for applications at hand than the covariant form pioneered by Avramidi and adapted by Henry-Labordère. The latter, being intrinsic, is preferable for higher order corrections. However, the Yoshida approach is completely self-contained and, especially when the coefficients in the diffusions depend explicitly on time, introduces some clear simplifications of the necessary computations.

Consider the following one dimensional parabolic differential equation

$$u_t + \mathcal{L}u = u_t + \frac{1}{2}a(s,t)^2u_{ss} + b(s,t)u_s + c(s,t)u = 0, \tag{3.1}$$

where subscripts refer to corresponding partial differentiations. In our case, $a(s,t) = s\sigma(s,t)$, where $\sigma(s,t)$ is the local volatility function. Note that $a(s,t)$ vanishes at $s = 0$ so it is not non-degenerate at this point. For the applicability of Yoshida's method in this case see Remark 3.5. We seek an expansion for the kth order approximation to the fundamental solution $p(s,t,K,T)$ in the form

$$p(s,t,K,T) \sim \frac{e^{-d(K,s,t)^2/2(T-t)}}{\sqrt{2\pi(T-t)}a(K,T)}\sum_{i=0}^{k} u_i(s,K,t)(T-t)^i, \tag{3.2}$$

where

$$d(K,s,t) = \int_K^s \frac{d\eta}{a(\eta,t)}$$

is the Riemannian distance between the points K and s with respect the time dependent Riemannian metric $ds^2/a(s,t)^2$. Yoshida [34] established that the coefficients u_i have the following form:

(3.3)
$$u_0(s,K,t) = \sqrt{\frac{a(s,t)}{a(K,t)}} \exp\left[-\int_K^s \frac{b(\eta,t)}{a(\eta,t)^2} d\eta - \int_K^s \frac{d_t(K,\eta,t)}{a(\eta,t)} d\eta\right].$$

and

$$u_i(s,K,t) = \frac{u_0(s,K,t)}{d(K,s,t)^i} \int_K^s \frac{d(K,\eta,t)^{i-1}}{u_0(\eta,K,t)} \left(\mathcal{L}u_{i-1} + \frac{\partial u_{i-1}}{\partial t}\right) \frac{d\eta}{a(\eta,t)}.$$
(3.4)

The function u_0 is given explicitly and u_i can be calculated recursively, and be performed in the mathematical software packages such as *Mathematica* or *Maple*. If $b = c = 0$ in the equation (3.1) and a is independent of time, we can calculate u_1 as follows:

$$\begin{aligned} u_1(s,K) &= \frac{u_0(s,K,t)}{4d(K,s)} \int_x^y \left(a''(\eta) - \frac{1}{2}\frac{a'(\eta)^2}{a(\eta)}\right) d\eta \\ &= \frac{1}{4d(K,s)} \sqrt{\frac{a(s)}{a(K)}} \left[a'(s) - a'(K) - \frac{1}{2}\int_K^s \frac{a'(\eta)^2}{a(\eta)} d\eta\right]. \end{aligned} \tag{3.5}$$

Here we need to use the explicit expression for u_0 mentioned earlier. In the general case with $b \neq 0$ and $c \neq 0$, we may compute an explicit expression for u_1 as follows. First, from above, we have that

$$u_1(s,K,t) = \frac{u_0(s,K,t)}{d(K,s,t)} \int_K^s \frac{1}{u_0(\eta,K,t)} \left[\frac{a^2}{2}\frac{\partial^2 u_0}{\partial s^2} + b\frac{\partial u_0}{\partial s} + cu_0 + \frac{\partial u_0}{\partial t}\right] \frac{d\eta}{a(\eta,t)}.$$

Recall that

$$\ln u_0 = \int_K^s \left[\frac{a_s(\eta,t)}{2} - \frac{b(\eta,t)}{a(\eta,t)} - d_t(K,\eta,t)\right] \frac{d\eta}{a(\eta,t)}.$$

For notational simplicity, we denote the integrand in the above integral by H, *i.e.*,

$$H(s,K,t) = \frac{\partial}{\partial s}\left[\ln u_0(s,K,t)\right] = \frac{a_s(s,t)}{2a(s,t)} - \frac{b(s,t)}{a(s,t)}^2 - \frac{d_t(K,s,t)}{a(s,t)}.$$

Then by straightforward computations, we can rewrite u_1 as

$$\frac{u_0(s,K,t)}{d(K,s,t)} \int_K^s \left[\frac{a^2}{2}\left(H^2 + H_s\right) + bH + c + \int_K^\eta H_t(\zeta,K,t) d\zeta\right] \frac{d\eta}{a(\eta,t)}$$
(3.6)

Remark 3.1. In the Black-Scholes setting, $a(s,t) = \sigma_{\rm BS} s$ and $u_0^{\rm BS}$ and $u_1^{\rm BS}$ are given explicitly as

$$u_0^{\rm BS}(s,K) = \sqrt{\frac{s}{K}}, \qquad u_1^{\rm BS}(s,K) = -\frac{\sigma_{BS}^2}{8}\sqrt{\frac{s}{K}}.$$

In fact, by straightforward computations, all the $u_k^{\rm BS}$'s can be calculated out and are given by

$$u_k^{\rm BS}(s,K) = \frac{(-1)^k}{k!}\left(\frac{\sigma_{\rm BS}^2}{8}\right)^k \sqrt{\frac{s}{K}}, \tag{3.7}$$

which in turn yields the following heat kernel expansion for Black-Scholes' transition probability density $p_{\rm BS}(s,K,t)$ as

$$p_{\rm BS}(s,K,t) = \frac{\exp\left[-\frac{(\ln s - \ln K)^2}{2\sigma_{\rm BS}^2 t}\right]}{\sqrt{2\pi t}\,\sigma_{\rm BS} K}\sqrt{\frac{s}{K}}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left(\frac{\sigma_{\rm BS}^2 t}{8}\right)^k. \tag{3.8}$$

This formula can also be verified directly.

3.2. **Calculating option prices.** We follow the approach adopted by Kusnetsov [23], who wrote his thesis under the direction of Claudio Albanese, and Henry-Labordere [18] based on the earlier work of Dupire [13], and by Derman and Kani [12], who used the following method to obtain the call prices directly from the probability density function without the requirement for a spatial integration. Unlike the method described in SECTION 1, the result can be obtained without using Laplace's method. Thus an additional approximation is avoided at this stage. The Carr-Jarrow formula in [9] (this formula was later exploited by Derman and Kani and by Dupire) for the call prices $C(s,K,t,T)$ reads

$$C(s,K,t,T) = (s-K)^+ + \frac{1}{2}\int_t^T a(K,u)^2 p(s,t,K,u)du. \tag{3.9}$$

In the present setup, we use the Yoshida expansion (3.2) for the heat kernel $p(s,t,K,u)$. This gives

$$\begin{aligned} &C(s,K,t,T) - (s-K)^+ \\ \sim\ & \frac{1}{2\sqrt{2\pi}}\sum_{i=0}^{k}\left(\int_t^T a(K,u)e^{-d(K,s,t)^2/2(u-t)}(u-t)^{i-\frac{1}{2}}du\right)u_i(s,K,t). \end{aligned}$$

This method was already used by Henry-Labordère in [18]. When we seek to determine the coefficient $\sigma_{\rm BS,2}$ in the expansion for the *implied volatility* σ_{BS}, we will need to expand out (3.10), which we do in the following. We will show that

$$\begin{aligned}(3.10)\quad & \left[C(s,t,K,T)-(s-K)^+\right]e^{d(K,s,t)^2/2(T-t)} \\ = \; & C^{(1)}(s,K,t)(T-t)^{3/2}+C^{(2)}(s,K,t)(T-t)^{5/2}+o(T-t)^{5/2}.\end{aligned}$$

The key step in making the approach in SECTION 2 rigorous is to show that the remainder in (3.10) truly is $o(T-t)^{5/2}$. This in turn will follow, as explained in from the fact that the first few terms in the "geometric series expansion" can be complemented by the Levi parametrix method to ensure that we have

$$p(s,t,K,T)=\frac{e^{-\frac{d(K,s,t)^2}{2(T-t)}}}{\sqrt{2\pi}a(K,T)}\left[\sum_{i=0}^{2}u_i(s,K,t)(T-t)^{i-\frac{1}{2}}+o(T-t)^{\frac{3}{2}}\right],$$

i.e., that the suitably modified preliminary approximation of the heat kernel can actually give a convergent series with a tail of order smaller than the last term in the geometric series. Note that the theory requires us to proceed till order $k>n/2$ in the series (i.e., order 1 in the present case $n=1$) if all we want is the transition probability. If we wish to use the series to calculate a first order Greek, like the Delta, we would need to expand up to order 2, before using the Levi parametrix.

We now proceed with some additional approximations that will be necessary to obtain the expansion for the implied volatility. Note that

$$\begin{aligned}& \int_t^T a(K,u)e^{-\frac{d(K,s,t)^2}{2(u-t)}}(u-t)^{i-\frac{1}{2}}du \\ \sim \; & \int_t^T [a(K,t)+a_t(K,t)(u-t)]e^{-\frac{d(K,s,t)^2}{2(u-t)}}(u-t)^{i-\frac{1}{2}}du \\ = \; & a(K,t)\int_t^T e^{-\frac{d(K,s,t)^2}{2(u-t)}}(u-t)^{i-\frac{1}{2}}du \\ & \qquad +a_t(K,t)\int_t^T e^{-\frac{d(K,s,t)^2}{2(u-t)}}(u-t)^{i+\frac{1}{2}}du \\ = \; & a(K,t)U_i(d(K,s,t),T-t)+a_t(K,t)U_{i+1}(d(K,s,t),T-t),\end{aligned}$$

where we have introduced the function

$$U_i(\omega,\tau)=\int_0^\tau u^{i-\frac{1}{2}}e^{-\frac{\omega^2}{2u}}du \qquad (3.11)$$

for notational simplicity. Inserting this into (3.10) we get

(3.12)

$$\begin{aligned} &C(s,K,t,T)-(s-K)^+\\ \sim&\frac{1}{2\sqrt{2\pi}}\sum_{i=0}^{k}\left[a(K,t)U_i(d,T-t)+a_t(K,t)U_{i+1}(d,T-t)\right]u_i(s,K,t). \end{aligned}$$

Remark 3.2. The corresponding term in the Black-Scholes setting is

$$C_{\mathrm{BS}}(s,K,t,T)-(s-K)^+\sim\frac{\sqrt{sK}}{2\sqrt{2\pi}}\left[\sigma_{\mathrm{BS}}U_0(d_{\mathrm{BS}},T-t)-\frac{\sigma_{\mathrm{BS}}^3}{8}U_1(d_{\mathrm{BS}},T-t)\right],$$

where

$$d_{\mathrm{BS}}=d_{\mathrm{BS}}(K,s)=\frac{1}{\sigma_{\mathrm{BS}}}\ln\frac{s}{K}$$

is the distance between K and s in the Black-Scholes' setting. In fact, the complete series can be obtained by using the general formula (3.8) in Remark 3.1 and we have

$$C_{\mathrm{BS}}(s,K,t,T)-(s-K)^+=\frac{\sqrt{sK}\sigma_{\mathrm{BS}}}{2\sqrt{2\pi}}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left(\frac{\sigma_{\mathrm{BS}}^2}{8}\right)^k U_k(d_{\mathrm{BS}},T-t).$$

The leading order in the expression for the call price away from the money is $(T-t)^{3/2}e^{-d(K,s,t)^2/2(T-t)}$. At this point we are seeking the contributions up to order $(T-t)^{5/2}$. Canceling the common factor we have arrived at the following balance relation between the call prices from the local volatility model and from the Black-Scholes setting:

$$\begin{aligned} &\sqrt{sK}\left[\sigma_{\mathrm{BS}}U_0(d_{\mathrm{BS}},T-t)-\frac{\sigma_{\mathrm{BS}}^3}{8}U_1(d_{\mathrm{BS}},T-t)\right]\\ \sim\;&\left[a(K,t)U_0(d,T-t)+a_t(K,t)U_1(d,T-t)\right]u_0(s,K,t)\\ &+a(K,t)U_1(d,T-t)u_1(s,K,t). \end{aligned} \tag{3.13}$$

We now consider two regimes separately.

REGIME 1: $s\neq K$ positive and fixed and $T-t\downarrow 0$. We shall use the following asymptotic formulas for U_0 and U_1 that are easily obtained from the well known asymptotic formula for the complementary error function. As $\tau \to 0^+$, we have

$$\begin{aligned} U_0(\omega,\tau) &= 2\sqrt{\tau}e^{-\frac{\omega^2}{2\tau}} - 2\omega\int_{\frac{\omega}{\sqrt{\tau}}}^{\infty} e^{-\frac{x^2}{2}}dx \\ &= 2\left[\frac{\tau^{3/2}}{\omega^2} - \frac{3\tau^{5/2}}{\omega^4} + o\left(\tau^{5/2}\right)\right]e^{-\frac{\omega^2}{2\tau}}, \\ U_1(\omega,\tau) &= \frac{2\tau^{3/2}}{3}e^{-\frac{\omega^2}{2\tau}} - \frac{\omega^2}{3}U_0(\omega,\tau) \\ &= \left[\frac{2\tau^{5/2}}{\omega^2} + o\left(\tau^{5/2}\right)\right]e^{-\frac{\omega^2}{2\tau}} \end{aligned}$$

These expansions will be applied to $\omega = d(K,s,t)$ (local volatility) and $\omega = d_{\mathrm{BS}}(K,s)$. We now let

$$\xi = \ln\frac{s}{K}.$$

The relation between d_{BS} and σ_{BS} is

$$d_{\mathrm{BS}} = \frac{\xi}{\sigma_{\mathrm{BS}}}. \tag{3.14}$$

In the time inhomogeneous case we seek an expansion for σ_{BS} in the form

$$\sigma_{\mathrm{BS}}(t,T) = \sigma_{\mathrm{BS},0}(t) + \sigma_{\mathrm{BS},1}(t)(T-t) + \sigma_{\mathrm{BS},2}(t)(T-t)^2 + o\left((T-t)^2\right).$$

Note the dependence of the coefficients in the expansion on the *spot variable* t. This dependence is absent in the time homogeneous case, as will be clear from the result of the expansion. Its presence in the case of time inhomogeneous diffusions is natural since already the transition probability density depends jointly on t and T and not only on their difference. We seek natural expressions for the coefficients which do not depend on the expiry explicitly. Mathematically it is of course also possible to expand around $t = T$, but in this case more terms are needed to recover the same accuracy.

Now using the above expansions for $U_0(\omega,\tau)$ and $U_1(\omega,\tau)$ and relation (3.14) we see that the left hand side of (3.13) becomes

$$\begin{aligned}(3.15)\quad &\frac{\sqrt{sK}}{\xi^2}e^{-\frac{\xi^2}{2\sigma_{\mathrm{BS},0}^2(T-t)}+\frac{\xi^2\sigma_{\mathrm{BS},1}}{\sigma_{\mathrm{BS},0}^3}}\\ &\quad\left[\sigma_{\mathrm{BS},0}^3+\left(3\sigma_{\mathrm{BS},0}^2\sigma_{\mathrm{BS},1}-\frac{\xi^2\sigma_{\mathrm{BS},0}}{2}\left[3\left(\frac{\sigma_{\mathrm{BS},1}}{\sigma_{\mathrm{BS},0}}\right)^2-\frac{2\sigma_{\mathrm{BS},2}}{\sigma_{\mathrm{BS},0}}\right]\right.\right.\\ &\qquad\left.\left.-\sigma_{\mathrm{BS},0}^5\left[\frac{3}{\xi^2}+\frac{1}{8}\right]\right)(T-t)\right].\end{aligned}$$

Here we have used the following expansion for expanding the exponent in the exponential term:

$$\frac{1}{\sigma_{\mathrm{BS}}^2}\sim\frac{1}{\sigma_{\mathrm{BS},0}^2}\left[1-\frac{2\sigma_{\mathrm{BS},1}}{\sigma_{\mathrm{BS},0}}(T-t)+\left(\frac{3\sigma_{\mathrm{BS},1}^2}{\sigma_{\mathrm{BS},0}^2}-\frac{2\sigma_{\mathrm{BS},2}}{\sigma_{\mathrm{BS},0}}\right)(T-t)^2\right].$$

On the local volatility side, on the other hand, we have, after again canceling a factor $(T-t)^{3/2}$ and a factor $\frac{1}{2\sqrt{2\pi}}$, that the terms of up to order $T-t$ are

$$\begin{aligned}(3.16)\quad &e^{-\frac{d^2}{2(T-t)}}\\ &\left[\frac{a(K,t)u_0}{d^2}+\left(\frac{a_t(K,t)u_0}{d^2}-\frac{3a(K,t)u_0}{d^4}+\frac{a(K,t)u_1}{d^2}\right)(T-t)\right].\end{aligned}$$

Finally, from the corresponding terms in (3.15) and (3.16),

- by matching the exponential term, we obtain

$$(3.17)\qquad \sigma_{\mathrm{BS},0}=\frac{\xi}{d(K,s,t)}=\frac{\ln\left(\frac{s}{K}\right)}{\int_K^s\frac{d\eta}{a(\eta,t)}}.$$

- by matching the constant term, we obtain

$$(3.18)\quad \sigma_{\mathrm{BS},1}=\frac{\sigma_{\mathrm{BS},0}^3\ln\left[\frac{u_0(s,K,t)a(K,t)\xi^2}{\sqrt{sK}d^2\sigma_{\mathrm{BS},0}^3}\right]}{\xi^2}=\frac{\xi\ln\left[\frac{a(K,t)u_0(s,K,t)d(K,s,t)}{\xi\sqrt{sK}}\right]}{d(K,s,t)^3}.$$

- by matching the first order term, we obtain

$$\sigma_{\mathrm{BS},2} = -\frac{3\sigma_{\mathrm{BS},1}\sigma^2_{\mathrm{BS},0}}{\xi^2} + \frac{3\sigma^2_{\mathrm{BS},1}}{2\sigma_{\mathrm{BS},0}} + \frac{\sigma^3_{\mathrm{BS},0}}{\xi^2}\left[\frac{3\sigma^2_{\mathrm{BS},0}}{\xi^2} + \frac{\sigma^2_{\mathrm{BS},0}}{8} + \frac{a_t(K,t)}{a(K,t)} - \frac{3}{d^2(K,s,t)} + \frac{u_1(s,K,t)}{u_0(s,K,t)}\right]$$
$$= -\frac{3\sigma_{\mathrm{BS},1}}{d^2} + \frac{3\sigma^2_{\mathrm{BS},1}}{2\sigma_{\mathrm{BS},0}} + \frac{\xi^3}{8d^5} + \frac{\xi}{d^3}\left[\frac{a_t(K,t)}{a(K,t)} + \frac{u_1(s,K,t)}{u_0(s,K,t)}\right]. \tag{3.19}$$

Above we need the expression for u_1 obtained earlier, see equation (3.6).

REGIME 2: $s = K > 0$ (at the money) and $T - t \downarrow 0$. We use again the expansion (3.12) for the call price, after setting $\tau = T - t$ and $s = K$

$$C(K,K,t,T) \sim \frac{1}{2\sqrt{2\pi}} \sum_{i=0}^{\infty} [aU_i(0,\tau) + a_t U_{i+1}(0,\tau)]\, u_i(K,K,t),$$

and this time we keep the terms up to order $\tau^{\frac{5}{2}}$. Trivially, with $\omega = 0$ in (3.11) we have

$$U_0(0,\tau) = 2\sqrt{\tau}, \quad U_1(0,\tau) = \frac{2}{3}\tau^{\frac{3}{2}}, \quad U_2(0,\tau) = \frac{2}{5}\tau^{\frac{5}{2}}.$$

On the Black-Scholes side we then have, after dropping the factor $\frac{1}{2\sqrt{2\pi}}$,

$$K\left[2\left(\sigma_{\mathrm{BS},0} + \tau\sigma_{\mathrm{BS},1} + \tau^2\sigma_{\mathrm{BS},2}\right)\sqrt{\tau} - \frac{1}{12}\left(\sigma_{\mathrm{BS},0} + \sigma_{\mathrm{BS},1}\tau + \sigma_{\mathrm{BS},2}\tau^2\right)^3\tau^{\frac{3}{2}} + \frac{1}{320}\left(\sigma_{\mathrm{BS},0} + \sigma_{\mathrm{BS},1}\tau + \sigma_{\mathrm{BS},2}\tau^2\right)^5\tau^{\frac{5}{2}} + o(\tau^{\frac{5}{2}})\right]$$

On the local volatility side we have

$$2au_0\sqrt{\tau} + \frac{2}{3}(a_t u_0 + au_1)\tau^{\frac{3}{2}} + \frac{2}{5}(a_t u_1 + au_2)\tau^{\frac{5}{2}} + o(\tau^{\frac{5}{2}}),$$

where we have omitted the dependence on the independent variables, in all of which we replace s by K. Grouping the Black-Scholes contribution in powers of τ we obtain

$$2K\sigma_{\mathrm{BS},0}\sqrt{\tau} + K\left(\sigma_{\mathrm{BS},1} - \frac{1}{12}\sigma^3_{\mathrm{BS},0}\right)\tau^{\frac{3}{2}} + \left[2K\sigma_{\mathrm{BS},2} - \frac{K}{4}\sigma^2_{\mathrm{BS},0}\sigma_{\mathrm{BS},1} + K\frac{\sigma^5_{\mathrm{BS},0}}{320}\right]\tau^{\frac{5}{2}} + o(\tau^{\frac{5}{2}}).$$

Matching the coefficients of the powers of τ and using $u_0(K,K,t) = 1$, we obtain

$$\begin{aligned} &K\sigma_{\mathrm{BS},0} = a(K,t),\\ &K\left(2\sigma_{\mathrm{BS},1} - \frac{1}{12}\sigma^3_{\mathrm{BS},0}\right) = \frac{2}{3}(a_t + au_1),\\ &2K\sigma_{\mathrm{BS},2} - \frac{K}{4}\sigma^2_{\mathrm{BS},0}\sigma_{\mathrm{BS},1} + \frac{K}{320}\sigma^5_{\mathrm{BS},0} = \frac{2}{5}(a_t u_1 + au_2). \end{aligned}$$

They yield consecutively,

$$\begin{aligned} \sigma_{\mathrm{BS},0} &= \frac{a(K,t)}{K},\\ \sigma_{\mathrm{BS},1} &= \frac{a_t + au_1}{3K} + \frac{a^3(K,t)}{24K^3},\\ \sigma_{\mathrm{BS},2} &= \frac{a_t u_1 + au_2}{5K} + \frac{\sigma^2_{\mathrm{BS},0}\sigma_{\mathrm{BS},1}}{8} - \frac{\sigma^5_{\mathrm{BS},0}}{640}. \end{aligned}$$

Remark 3.3. We have checked by explicit computation that in the time-homogeneous case, these at-the-money expressions for the $\sigma_{\mathrm{BS,i}}$, $i = 0,1,2$, coincide with limits as $s \to K$ of the out-of-the-money expressions (3.17), (3.18) and (3.19).

Remark 3.4. Taking into account non-zero interest rates
It is straightforward to combine the Yoshida approach with non-zero interest rates and/or dividends to account for the presence of the r dependent term in (2.6). Note that if the stock satisfies the time homogeneous SDE: $dS_t = rS_t dt + S_t\sigma(S_t)dW_t$, call it Problem (I), then the forward price $f_t = e^{r(T-t)}S_t$ satisfies the driftless but time inhomogeneous SDE: $df_t = f_t\sigma\left(e^{-r(T-t)}f_t\right)dW_t = f_t\tilde{\sigma}(f_t,t)dW_t$, call it Problem (II). The relationship between the implied volatility σ^f_{BS} for problem (II) and that for problem (I) σ^r_{BS} is easily seen to be

$$\sigma^r_{\mathrm{BS}}(s,t,K,T) = \sigma^f_{\mathrm{BS}}\left(s,t,Ke^{-r(T-t)},T\right).$$

From this it follows that

$$\sigma^r_{\mathrm{BS},0}(s,K) = \sigma^f_{\mathrm{BS},0}(s,K) = \frac{\log(s/K)}{\int_x^y \frac{du}{u\sigma(u)}},$$

$$\sigma^r_{\mathrm{BS},1}(s,K) = \sigma^f_{\mathrm{BS},1}(s,K) + \frac{\partial}{\partial t}\left[\sigma^f_{\mathrm{BS},0}\left(s, Ke^{r(T-t)}\right)\right]\Bigg|_{t=T}$$

$$= \sigma^f_{\mathrm{BS},1}(s,K) + \frac{r}{\int_K^s \frac{du}{u\sigma(u)}} - \frac{\frac{r\log(s/K)}{\sigma(K)}}{\left(\int_K^s \frac{du}{u\sigma(u)}\right)^2}, \tag{3.20}$$

where in the first term above we need to determine $\sigma^f_{\mathrm{BS},1}(y,x)$ for what is now a time inhomogeneous problem (II). Calculating this expression we find, since the volatility $\tilde{\sigma}$ now depends *explicitly* on time, that compared to (3.18) there are additional r-dependent terms which are given by:

$$\frac{-r\int_x^y \frac{1}{u\sigma^2(u)}du + \frac{r}{\sigma(K)}\int_x^y \frac{1}{u\sigma(u)}}{\frac{(\int_K^s \frac{1}{u\sigma(u)}du)^3}{\log(s/K)}}$$

$$= \frac{-r\int_K^s \frac{1}{u\sigma^2(u)}du}{\frac{(\int_K^s \frac{1}{u\sigma(u)}du)^3}{\log(s/K)}} + \frac{\frac{r\log(s/K)}{\sigma(K)}}{(\int_K^s \frac{1}{u\sigma(u)})^2}$$

The second term in the above expression cancels the third term in (3.20) to produce exactly the expression (2.6).

The procedure we have outlined above that allows us, with a few calculations, to pass from the zero to the non zero interest case, can be repeated to determine the second order correction.

Remark 3.5. Yoshida's approach requires some global hypothesis on the coefficients of the parabolic equation. In particular, it requires the equation to be non-degenerate:

$$\min_{t\in[0,T],S\in\mathbb{R}} a(S,t) = c > 0.$$

Models like the CEV model, which will be considered in the numerical section, and even the Black-Scholes model itself, do not satisfy this condition since they are degenerate when $S = 0$. There may also be problems at $S = \infty$. We have encountered similar problem in the probabilistic approach in SECTION 2. However, as we have explained in SECTION 2, as long as we keep away from these two boundary points, the behavior of the coefficient functions in a neighborhood of the boundary points are irrelevant as long as we also impose some moderate conditions on the growth of $a(S,t)$. In particular the call price expansion will not be affected. This principle of not feeling the boundary is explained in detail in APPENDIX A.

4. NUMERICAL RESULTS

To recap, we have derived an expansion formula for implied volatility up to second order in time-to-expiration in the form

$$\sigma_{\mathrm{BS}}(t,T) = \sigma_{\mathrm{BS},0}(t) + \sigma_{\mathrm{BS},1}(t)(T-t) + \sigma_{\mathrm{BS},2}(t)(T-t)^2 + o\left((T-t)^2\right).$$

where the coefficients $\sigma_{\mathrm{BS,i}}(t)$ are given by (3.17), (3.18) and (3.19).

To test this expansion formula numerically, we use well-known exact formulae for option prices in two specific time-homogeneous local volatility models: the CEV model and the quadratic local volatility model, as developped by Lipton [27], Zuhlsdorff [33], Andersen [1] and others.

Time dependence is modeled as a simple time-change so that these exact time-independent solutions may be re-used. Specifically, the time-change is:

$$\tau(T) = \int_0^T e^{-2\lambda t}\,dt = \frac{1}{2\lambda}\left(1 - e^{-2\lambda T}\right)$$

Throughout, for simplicity, we assume zero interest rates and dividends so that $b(s,t) = 0$ in equation (3.1).

4.1. The Henry-Labordère approximation.

Pierre Henry-Labordère presents a heat kernel expansion based approximation to implied volatility in equation (5.40) on page 140 of his book [19]:

(4.1)
$$\sigma_{BS}(K,T) \approx \sigma_0(K,t)\left\{1 + \frac{T}{3}\left[\frac{1}{8}\sigma_0(K,t)^2 + \mathcal{Q}(f_{av}) + \frac{3}{4}\mathcal{G}(f_{av})\right]\right\}$$

with

$$\mathcal{Q}(f) = \frac{C(f)^2}{4}\left[\frac{C''(f)}{C(f)} - \frac{1}{2}\left(\frac{C'(f)}{C(f)}\right)^2\right]$$

and

$$\mathcal{G}(f) = 2\,\partial_t \log C(f) = 2\,\frac{\partial_t a(f,t)}{a(f,t)}$$

where $C(f) = a(f,t)$ in our notation, $f_{av} = (s+K)/2$ and the term $\sigma_0(K,t)$ is our lowest order coefficient (3.17) originally derived in [4]:

$$\sigma_0(K,t) = \frac{\log(s/K)}{\int_K^s \frac{d\eta}{a(\eta\, t)}}$$

On page 145 of his book, Henry-Labordère presents an alternative approximation to first order in $T-t$, matching ours exactly in the time-homogeneous case and differing only slightly in the time-inhomogeneous case. In Section 2, we demonstrated that our approximation is the optimal one to first order in $T-t$.

4.2. **Model definitions and parameters.**

4.2.1. *CEV model.* The SDE is

$$df_t = e^{-\lambda\, t}\, \sigma\, \sqrt{f_t}\, dW_t$$

with $\sigma = 0.2$. In the time-independent version, $\lambda = 0$ and in the time-dependent version, $\lambda = 1$. For the CEV model therefore,

$$\mathcal{Q}(f) = -\frac{3}{32}\,\frac{\sigma^2}{f}$$

and

$$\mathcal{G}(f) = -2\,\lambda$$

so the Henry-Labordère approximation (4.1) becomes

$$\sigma_{BS}(K,T) \approx \sigma_0(K,t)\, \left\{1 + \frac{T}{3}\left[\frac{1}{8}\,\sigma_0(K,t)^2 - \frac{3}{32}\,\frac{\sigma^2}{f_{av}} - \frac{3}{2}\,\lambda\right]\right\}$$

The closed-form solution for the square-root CEV model is well-known and can be found, for example, in Shaw[30].

4.2.2. *Quadratic model.* The SDE is

$$df_t = e^{-\lambda\, t}\, \sigma\, \left\{\psi\, f_t + (1-\psi) + \frac{\gamma}{2}\,(f_t - 1)^2\right\}\, dW_t$$

with $\sigma = 0.2$, $\psi = -0.5$, and $\gamma = 0.1$. Again in the time-independent version, $\lambda = 0$ and in the time-dependent version, $\lambda = 1$. Then for the quadratic model,

$$\begin{aligned}\mathcal{Q}(f) = &\frac{1}{32}\,\sigma^2\, \Big\{(f-1)^3\, (3\,f+1)\, \gamma^2 + 24\,(1-\psi)\,\gamma\, f \\ &+ 12\,\psi\,\gamma\, f^2 - 4\, \left[(4-3\,\psi)\,\gamma + \psi^2\right]\Big\}\end{aligned}$$

and again

$$\mathcal{G}(f) = -2\,\lambda$$

The closed-form solution for the quadratic model with these parameters[1] is given in Andersen[1].

4.3. **Results.** In TABLES 1 and 2, respectively, we present the errors in the above approximations in the case of time-independent CEV and quadratic local volatility functions (with $\lambda = 0$ and $T = 1$). We note that our approximation does slightly better than Henry-Labordère's, although the errors in both approximations are negligible. In Figures 1 and 2 respectively, these errors are plotted.

TABLE 1. CEV model implied volatility errors for various strike prices in the Henry-Labordère (HL) approximation and our first and second order approximations respectively. The exact volatility in the last column is obtained by inverting the closed-form expression for the option price in the CEV model.

Strikes	$\Delta\sigma_{HL}$	$\Delta\sigma_1$	$\Delta\sigma_2$	σ_{exact}
0.50	2.12e-05	1.31e-06	1.98e-08	0.2368
0.75	3.46e-06	7.98e-07	9.87e-09	0.2148
1.00	5.68e-07	5.68e-07	6.03e-09	0.2001
1.25	1.52e-06	4.21e-07	4.08e-09	0.1891
1.50	3.45e-06	3.33e-07	2.96e-09	0.1805
1.75	5.45e-06	2.73e-07	2.18e-09	0.1734
2.00	7.27e-06	2.29e-07	1.70e-09	0.1674

In Figures 3 and 4, we plot results for the time-dependent cases $\lambda = 1$ with $T = 0.25$ and $T = 1.0$ respectively, comparing our approximation to implied volatility with the exact result. To first order in $\tau = T - t$ (with only σ_1 and not σ_2), we see that our approximation is reasonably good for short expirations ($\lambda T \ll 1$) but far off for longer expirations ($\lambda T > 1$). The approximation including σ_2 up to order τ^2 is almost exact for the shorter expiration $T = 0.25$ and much closer to the true implied volatility for the longer expiration $T = 1.0$.

[1]The solution is more complicated for certain other parameter choices

TABLE 2. Quadratic model implied volatility errors for various strike prices in the Henry-Labordère (HL) approximation and our first and second order approximations respectively. The exact volatility in the last column is obtained by inverting the closed-form expression for the option price in the quadratic model.

Strikes	$\Delta\sigma_{HL}$	$\Delta\sigma_1$	$\Delta\sigma_2$	σ_{exact}
0.50	-8.83e-05	-1.04e-05	-1.08e-07	0.3129
0.75	-3.42e-05	-3.05e-06	-1.94e-08	0.2451
1.00	-2.14e-06	-1.09e-06	-4.58e-09	0.2003
1.25	1.99e-05	-4.31e-07	-1.30e-09	0.1675
1.50	3.32e-05	-1.80e-07	-3.92e-10	0.1418
1.75	4.13e-05	-7.59e-08	-5.28e-11	0.1209
2.00	4.56e-05	-3.16e-08	9.57e-12	0.1032

APPENDIX A. PRINCIPLE OF NOT FEELING THE BOUNDARY

Consider a one-dimensional diffusion process

$$dX_t = a(X_t)\,dW_t + b(X_t)\,dt$$

on $[0,\infty)$, where the continuous function $a(x) > 0$ for $x > 0$. We assume that b is continuous on $\mathbb{R}_+$. We do not make any assumption about the behavior of $a(y)$ as $y \downarrow 0$. Let $d(a,b)$ be the distance between two points $a, b \in \mathbb{R}_+$ determined by $1/a$. If say $a < b$, then

$$d(a,b) = \int_a^b \frac{dx}{a(x)}.$$

Let

$$\tau_c = \inf\{t \geq 0 : \ X_t = c\}.$$

Lemma A.1. *Suppose that $x > 0$ and $c > 0$. Then*

$$\lim_{\tau\downarrow 0} \tau \ln \mathbb{P}_x \{\tau_c \leq \tau\} \leq -\frac{d(x,c)^2}{2}.$$

Proof. Let $Y_t = d(X_t, x)$. By Itô's formula we have

$$dY_t = dW_t + \theta(Y_t)\,dt,$$

where

$$\theta(y) = \frac{b(z)}{a(z)} - \frac{1}{2}a'(z), \qquad y = d(z,x).$$

Without loss of generality we assume that $c > x$. It is clear that

$$\mathbb{P}_x\left\{\tau_c^X \leq \tau\right\} = \mathbb{P}_0\left\{\tau_D^Y \leq \tau\right\}, \qquad D = d(x,c).$$

Let θ be the lower bound of the function $\theta(z)$ on the interval $[0, D]$. Then $Y_t \le D$ for all $0 \le t \le \tau$ implies that $W_t \le D - \theta\tau$ for all $0 \le t \le \tau$. It follows that

$$\mathbb{P}_0 \left\{ \tau_D^Y \le \tau \right\} \le \mathbb{P}_0 \left\{ \tau_{D-\theta\tau}^W \le \tau \right\}.$$

The last probability is explicitly known:

$$\mathbb{P}_0 \left\{ \tau_\lambda^W \le \tau \right\} = \frac{\lambda}{\sqrt{2\pi}} \int_0^\tau t^{-3/2} e^{-\lambda^2/2t}\, dt.$$

Using this we have after some routine manipulations

$$\lim_{\tau\downarrow 0} \tau \ln \mathbb{P}_0 \left\{ \tau_{D-\theta\tau} \le \tau \right\} \le -\frac{D^2}{2}.$$

The desired result follows immediately.

Note that we do not need to assume that X does not explode. By convention $\tau_c = \infty$ if X explodes before reaching c, thus making the inequality more likely to be true. □

Let $0 < a < x < b < \infty$. Let f be a nonnegative function on $\mathbb{R}_+$ and suppose that f is supported on $x \ge b$, i.e., $f(y) = 0$ for $y \le b$. This corresponds to the case of an out-of-the-money call option. Consider the call price

$$v(x, \tau) = \mathbb{E}_x f(X_\tau).$$

We compare this with

$$v_1(x, \tau) = \mathbb{E}_x \left\{ f(X_\tau); \tau < \tau_a \right\}.$$

Note that v_1 only depends on the values of a on $[a, \infty)$, thus he behavior of a near $y = 0$ is excluded from consideration. We have

$$v(x, \tau) - v_1(x, \tau) = \mathbb{E}_x \left\{ f(X\tau); \tau_a \le \tau \right\} \stackrel{\text{def}}{=} v_2(x, \tau).$$

By the Markov property we have

$$v_2(x, \tau) = \mathbb{E}_x \left\{ \mathbb{E}_a f(X_s)|_{s=\tau-\tau_a}; \tau_a \le \tau \right\}.$$

Now since $f(y) = 0$ for $y \le b$, we have

$$\mathbb{E}_a f(X_s) = \mathbb{E}_a \left\{ f(X_s); \tau_b \le s \right\}.$$

Using the Markov property again we have

$$\mathbb{E}_a f(X_s) = \mathbb{E}_a \left\{ \mathbb{E}_b f(X_t)|_{t=s-\tau_b}; \tau_b \le s \right\}.$$

We assume that

$$\sup_{0\le t\le 1} \mathbb{E}_b f(X_t) \le C.$$

This assumption satisfied if we bound the growth rates of f, a and b at infinity appropriately. A typical case is when f grows exponentially (call option), and a and b grow at most linearly. These conditions are satisfied by all the popular models we deal with. It is clear that we have to make some assumption about the behavior of the data at infinity, otherwise the problem may not even make any sense. Under this hypothesis we have

$$\mathbb{E}_a f(X_s) \leq C\mathbb{P}_a \{\tau_b \leq s\}.$$

Now we have

$$v_2(x,\tau) \leq C\mathbb{P}_a \{\tau_b \leq \tau\}.$$

It follows from the LEMMA that

$$\lim_{\tau\downarrow 0} \tau \ln v_2(x,\tau) \leq -\frac{d(a,b)^2}{2}.$$

Recall that

$$v(x,\tau) = v_1(x,\tau) + v_2(x,\tau).$$

The function $v_1(x,\tau)$ does not depend on the values of a near $y = 0$. We can alter the values of a near $y = 0$ and the resulting error is bounded asymptotically by $\exp\left[-d(a,b)^2/2\tau\right]$. Now if the support of f (as a closed set) contains $y = b$, then we can prove, assuming a behaves nicely near $y = 0$ if necessary, that

$$\lim_{\tau\downarrow 0} \tau \ln \mathbb{E}_x f(X_\tau) = -\frac{d(x,b)^2}{2}.$$

Since $d(x,b) < d(a,b)$, we have proved the following principle of not feeling the boundary.

Theorem A.2. *Let X^1 and X^2 be (a_1,b_1)- and (a_2,b_2)-diffusion processes on $\mathbb{R}_+$, respectively, f a nonnegative function on $\mathbb{R}_+$, and $0 < a < x < b$. Suppose that a_i, b_i, f satisfy the conditions stated above. Suppose further that $a_1(y) = a_2(y)$ for $y \geq a$. Then*

$$\limsup_{\tau\downarrow 0} \tau \ln \left|\mathbb{E}_x f(X^1_\tau) - \mathbb{E}_x f(X^2_\tau)\right| \leq -\frac{d(a,b)^2}{2}$$

and

$$\lim_{\tau\downarrow 0} \tau \ln \mathbb{E}_x f(X^i_\tau) = -\frac{d(x,b)^2}{2}.$$

Corollary A.3. *Under the same conditions, we have*

$$\lim_{\tau\downarrow 0} \frac{\mathbb{E}_x f(X^1_\tau)}{\mathbb{E}_x f(X^2_\tau)} = 1.$$

See Hsu [21] for a more general principle of not feeling the boundary for higher dimensional diffusions.

References

[1] ANDERSEN, L.B.G., Option pricing with quadratic volatility: A revisit, Discussion paper, *Banc of America Securities*, 2008.

[2] ANDREASEN, J., and ANDERSEN, L.B.G., Jump diffusion pricing: volatility smile fitting and numerical methods for option pricing, *Review of Derivatives Research*, Vol 4, pp.231-262, 2000.

[3] BATES, D., Jumps and stochastic volatility: the exchange rate processes implicit in Deutschemark opions, *The Review of Financial Studies*, Vol 9, pp.69-107, 1996.

[4] BERESTYCKI, H., BUSCA, J., and FLORENT, I., Asymptotics and calibration of local volatility models, *Quantitative Finance*, Vol 2, pp.61-69, 2002.

[5] BERESTYCKI, H., BUSCA, J., and FLORENT, I., Computing the implied volatility in stochastic volatility models, *Communications on Pure and Applied Mathematics*, Vol 57, no.10, pp.1352-1373, 2004.

[6] BOURGADE, P., and CROISSANT, O., Heat kernel expansion for a family of stochastic volatility models: δ geometry, *Preprint*, available at SSRN, 2005.

[7] DE BRUIN, N. G., Asymptotic Methods in Analysis, *Dover Publications*, 1999.

[8] CARR, P., GEMAN, H., MADAN, D., and YOR, M., From Local volatility to local Lévy models, *Quantitative Finance*, Vol 4, pp.581-588, 2004.

[9] CARR, P. and JARROW, R., The stop-loss start-gain paradox and option valuation: A new decomposition into intrinsic and time value, *Review of Financial Studies*, Vol 3, no.3, pp.469-492, 1990.

[10] CARR, P., and WU, L., Stochastic skew for currency options, *Journal of Financial Economics*, Vol 86, no.1, pp.213-247, 2007.

[11] CHAVEL, I., Eigenvalues in Riemannian geometry, *Academic Press*, 1984.

[12] DERMAN, E., and KANI, I., Riding on a smile, *Risk Magazine*, Vol 7, pp.32-39, 1994.

[13] DUPIRE, B., A unified theory of volatility, Discussion paper Paribas Capital Management. Reprinted in "Derivative Pricing: the classic collection", edited by Peter Carr, *Risk Books*, 2004.

[14] GATHERAL, J., The volatility surface: a practitioner's guide, *Wiley Finance Series*, 2006.

[15] HAGAN, P. and WOODWARD, D., Equivalent Black volatilities, *Applied Mathematical Finance*, Vol 6, pp.147-157, 1999.

[16] HAGAN, P., LESNIEWSKI, A., and WOODWARD, D., Probability distribution in the SABR model of stochastic volatility, *Working Paper*, 2004.

[17] HAGAN, P., KUMAR, D., LESNIEWSKI, A., and WOODWARD, D., Managing Smile Risk, *Wilmott Magazine*, pp.84-108, 2003.

[18] HENRY-LABORDÈRE, P., A general asymptotic implied volatility for stochastic volatility models, *Preprint*, 2005.

[19] HENRY-LABORDÈRE, P., Analysis, geometry, and modeling in finance, *Chapman& Hall/CRC*, Financial Mathematics Series, 2008.

[20] HESTON, S., A closed form solutio for options with stochastoc volatility, with applications to bond and currency pricing, *The Review of Financial Studies*, Vol 6, pp.327-342, 1993.

[21] HSU, E. P., Principle of not feeling the boundary, *Indiana University Mathematics Journal*, Vol 39, no.2, pp.431-442, 1990.

[22] HULL, J. and WHITE, A., The pricing of options on assets with stochastic volatilities, *Journal of Finance*, Vol 42, no.2, pp.282-300, 1987.

[23] KUZNETSOV, A., PhD Thesis, Imperical College, 2004.

[24] KUNIMOTO, N. and TAKAHASHI, A., The asymptotic expansion approach to the valuation of interest rate contingent claims, *Mathematical Finance*, Vol 11, no.1, pp.117-151, 2001.

[25] LESNIEWSKI, A., WKB method for swaption smile, *Courant Institute Lecture*, 2002

[26] LESNIEWSKI A., Swaption Smiles via the WKB Method, *Mathematical Finance Seminar*, Courant Institute of Mathematical Sciences, 2002.

[27] LIPTON, A., The vol smile problem, *Risk Magazine*, February 2002.

[28] MEDVEDEV, A. and SCAILLET, O., Approximation and calibration of short-term implied volatilities under jump-diffusion stochastic volatility, *The Review of Financial Studies*, Vol 20, pp.427-459, 2007.

[29] MOLCHANOV, S., Diffusion processes and Riemannian geometry, *Russian Mathematical Surveys*, Vol 30, pp.11-63, 1975.

[30] SHAW, W.T., Modelling financial derivatives with Mathematica, *Cambridge University*, 1998.

[31] STEIN, E.M. and STEIN, J.C., Stock distributions with stochastic volatility: an analytic approach, *The Review of Financial Studies*, Vol 4, no.4, pp.727-752, 1991.

[32] TAKAHASHI, A., TAKEHARA, K. and TODA, M., Computation in an Asymptotic Expansion Method, University of Tokyo working paper CIRJE-F-621, 2009.

[33] ZUHLSDORFF, C., The pricing of derivatives on Assets with Quadratic Volatility, *Applied Mathematical Finance*, 8, pp. 235-262, 2001

[34] YOSHIDA, K., On the fundamental solution of the parabolic equation in a Riemannian space, *Osaka Mathematical Journal*, Vol 1, no. 1, 1953.

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FIGURE 1. Approximation errors in implied volatility terms as a function of strike price for the square-root CEV model with the parameters of Section 4.2.1. The solid line corresponds to the error in Henry-Labordère's approximation (5.40), and the dashed and dotted lines to our first and second order approximations respectively. Note that the error in our second order approximation is zero on this scale.

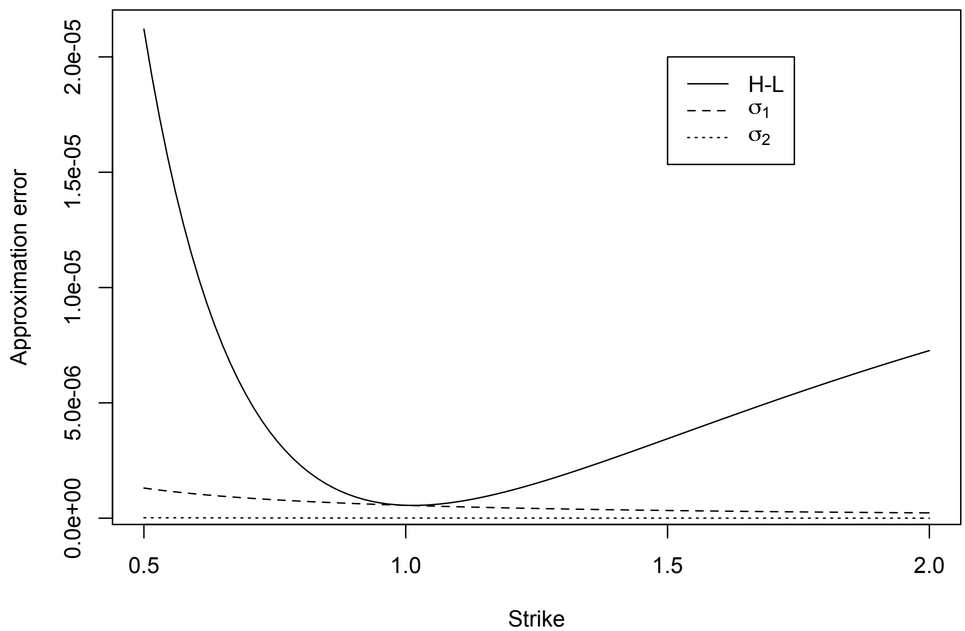

FIGURE 2. Approximation errors in implied volatility terms as a function of strike price for the quadratic model with the parameters of Section 4.2.2. The solid line corresponds to the error in Henry-Labordère's approximation (5.40), and the dashed and dotted lines to our first and second order approximations respectively. Note that the error in our second order approximation is zero on this scale.

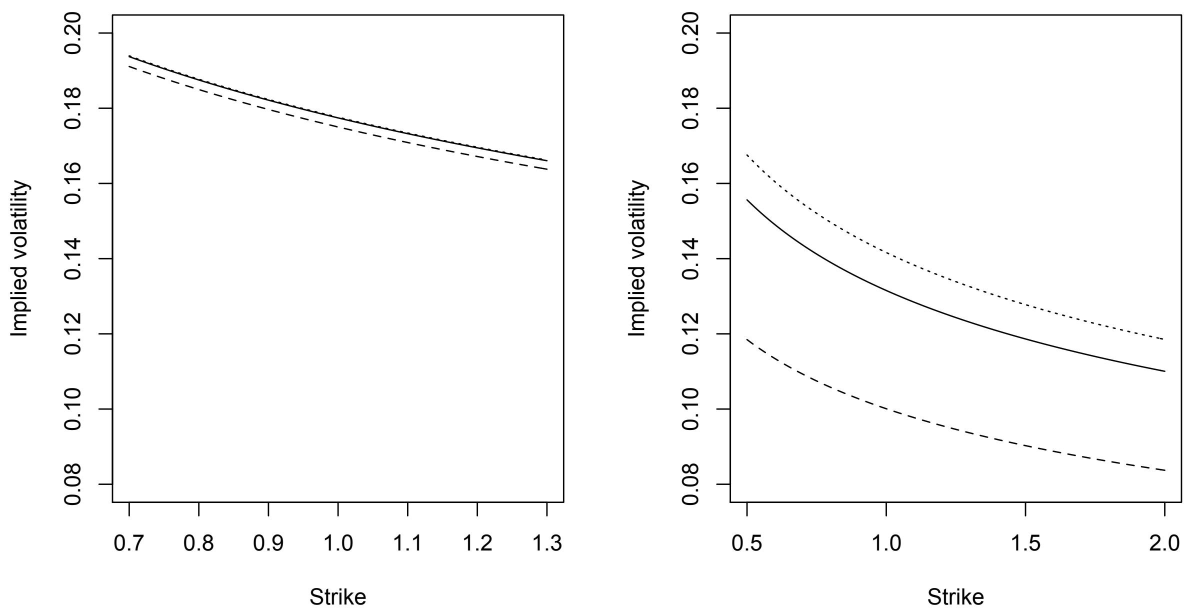

FIGURE 3. Implied volatility approximations in the CEV model with the parameters of Section 4.2.1 for two expirations: $\tau = 0.25$ on the left and $\tau = 1.0$ on the right. The solid line is exact implied volatility, the dashed line is our approximation to first order in $\tau = (T - t)$ (with only σ_1 and not σ_2) and the dotted line is our approximation to second order in τ (including σ_2).

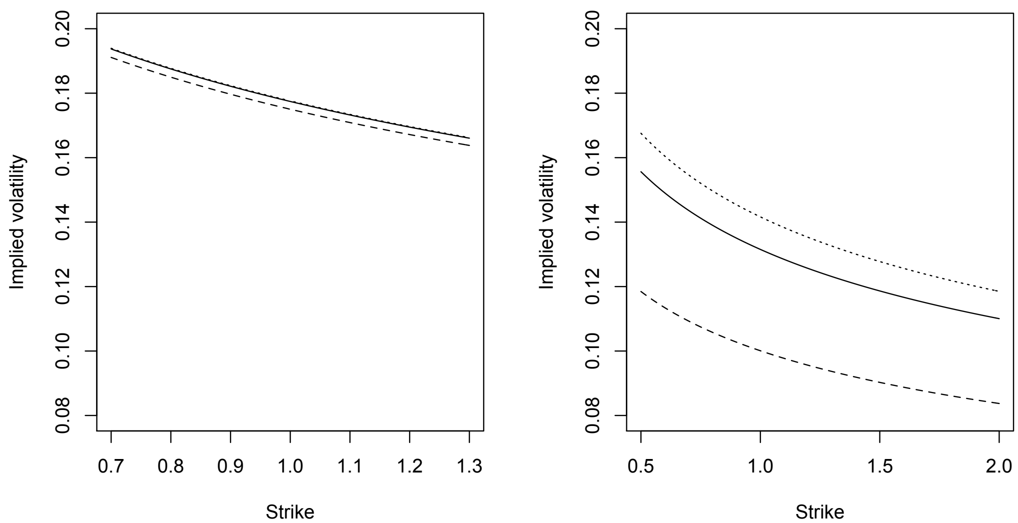

FIGURE 4. Implied volatility approximations in the quadratic model with the parameters of Section 4.2.2 for two expirations: $\tau = 0.25$ on the left and $\tau = 1.0$ on the right. The solid line is exact implied volatility, the dashed line is our approximation to first order in $\tau = (T - t)$ (with only σ_1 and not σ_2) and the dotted line is our approximation to second order in τ (including σ_2).